'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 3rd Semester Examination, 2021

CC5-MATHEMATICS

THEORY OF REAL FUNCTIONS AND INTRODUCTION TO METRIC SPACES

Time Allotted: 2 Hours | Full Marks: 60

The figures in the margin indicate full marks.
All symbols are of usual significance.

GROUP-A

1. Answer any ***four*** questions from the following: 3×4 = 12

(a) Show that the function f defined by $f(x)=\tan x$ is uniformly continuous over the closed interval $[a, b]$, where $-\frac{\pi}{2}<a<b<\frac{\pi}{2}$.

(b) Let $f(x)=\begin{cases}1, \text{ when } x \text{ is rational} \\ 0, \text{ when } x \text{ is irrational}\end{cases}$

State with reasons prove that f is discontinuous everywhere in $\mathbb{R}$.

(c) If f' exists on [0, 1], then show by Cauchy Mean value theorem that $f(1)-f(0)=\frac{1}{2x}f'(x)$ has at least one solution in (0, 1).

(d) Prove that the series,

$$\frac{1}{x+1}+\frac{x}{x+2}+\frac{x^2}{x+3}+\cdots(x>0)$$

converges if $0<x<1$, diverges if $x\geq 1$.

(e) Prove that the intersection of two open sets in a metric space is open.

(f) Give an example to show that every Cauchy sequence may not be a convergent sequence.

GROUP-B

2. Answer any ***four*** questions from the following: 6×4 = 24

(a) State and prove Taylor's theorem with Cauchy form of remainder. 6

(b) (i) Let a function f of x be uniformly continuous in the bounded open interval (a, b). Prove that $\lim_{x\to a+} f(x)$ exists finitely. 3

(ii) Prove that the function $f(x, y) = x^2 - 2xy + y^2 + x^3 - y^3 + x^5$ has neither a maximum nor a minimum at the origin. 3

(c) (i) Show that a real valued continuous function defined on a closed and bounded interval is uniformly continuous. 4

(ii) Determine the point of discontinuity of the function $f(x) = [\sin x]$ for all $x \in \mathbb{R}$ where $[x]$ denotes the integral part of x. 2

(d) Let $C[a, b]$ denote the set of all continuous function over the closed and bounded interval $[a, b]$. Consider the function $d : C[a, b] \times C[a, b] \to \mathbb{R}$ be given by 6

$$d(f, g) = \left(\int_a^b (f(x) - g(x))^2 dx \right)^{1/2}$$

check whether d is a metric on $C[a, b]$ or not.

(e) (i) Let A be any non-empty subset of metric space (X, d). Prove that the function $f : X \to \mathbb{R}$ defined by $f(x) = d(x, A)$ for all $x \in A$ is continuous. 3

(ii) Give an example of a metric on $\mathbb{R}$ for which the sequence $\{x_n\}_{n \in \mathbb{N}}$ where $x_n = n$ for all $n \in \mathbb{R}$ is convergent. 3

(f) Expand $\log(1 + x)$ for $x > -1$ in power of x, as an infinite series and mention the interval of validity of the expansion. 6

GROUP-C

3. Answer any ***two*** questions from the following: 12×2 = 24

(a) (i) If f is continuous in $[a, b]$ and $f(a) \cdot f(b) < 0$ then prove that there exists at least one c, where $a < c < b$ such that $f(c) = 0$. 6

(ii) Let f be a real valued function defined over $[-1, 1]$ such that 3+3

$$f(x) = \begin{cases} x \cos \frac{1}{x}, & \text{where } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$$

Does the Mean value theorem hold? $\lim_{x \to 0} f'(x)$ exist? Justify your answer.

(b) (i) Show that $\dfrac{\tan x}{x} > \dfrac{x}{\sin x}$, for $0 < x < \dfrac{\pi}{2}$ 6

(ii) Show that the minimum value of $\dfrac{(2x-1)(x-8)}{x^2 - 5x + 4}$ is greater than its maximum value. 6

(c) (i) Examine the differentiability of the function $f(x) = \sin[x]$ $\forall x \in \mathbb{R}$ on $\mathbb{R}$. 6

(ii) Prove that any two real roots of the equation $e^x \cos x + 1 = 0$ there is at least one real root of the equation $e^x \sin x + 1 = 0$. 3

(iii) Let X be any non-empty set and a distance function d on X is defined as 3

$$d(x, y) = 1, \text{ if } x \neq y$$
$$= 0, \text{ if } x = y.$$

Examine the sequence $\left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$ in (X, d) is convergent or not.

(d) (i) Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be two convergent sequences in metric space (X, d). Prove that the sequence $\{d(x_n, y_n)\}_{n\in\mathbb{N}}$ is a convergent sequence. 5

(ii) Give an example of function $f:\mathbb{R}\to\mathbb{R}$ which is continuous at $x = 1, 2, 3$ and discontinuous all the point except $x = 1, 2, 3$. Justify your answer. 4

(iii) A function $f:\mathbb{R}\to\mathbb{R}$ is continuous and $f(x)=0$ for all $x\in\mathbb{Q}$. Prove that $f(x)=0$ for all $x\in\mathbb{R}$. 3

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